Geometric Perspectives in Number Theory

By Benjamin Wagener, on Wednesday, March 19, 2014

The development of Algebraic Geometry offers nowadays very deep perspectives in Number Theory. It is such that we can expect that the future of Number Theory partially resides in this geometric development.

What is especially of interest is Arakelov Geometry and the general Arithmetic Geometric setting that allows a completely new change of view about Number Theory and more precisely about all what deals with Diophantine Mathematics.

Diophantine questions are some of the oldest of all Mathematics and are perhaps the most central of Number Theory. Those questions dealt with equations that are both the most elementary and the most general that are basically satisfied by numbers. What has remained nevertheless true for more than two thousand years is that those equations are in general impossible to solve.

The geometric settings about this are two folds. First those equations are reinterpreted as geometric objects for which an impressively wide theoretical framework is available thanks to algebraic geometry. Second and this is less well known, Arakelov Geometry and Arithmetic Geometry allow to turn everything about those questions and so about the most important questions about Numbers as geometric question. This may seem difficult to understand but this allows to “see” what is happening and this is a major step with regards to those questions that have been for long completely unsolvable.

In a whole, Number Theory may be turned completely geometric and this is in my opinion one of the biggest step that could ever be done in Number Theory.

For those who know Number Theory, the analytic side can be interpreted completely within the geometric context and the algebraic side completely fit inside the algebraic geometry settings. Modular forms and theta functions for example are naturally interpreted as sections of vector bundles above abelian varieties whereas the usual analytic tricks fit into a completely intrinsic setting with exact interpretations.

It seems not useless but perhaps necessary to insist on that fact that within the geometric setting everything becomes completely intrinsic.

The work about this has just begun and there are some general Diophantine results on linear groups and abelian varieties. Somehow the Mathematics of those objects is simplified because of various structures that make things easier to deal with and simplify the

structures. The major step will be done when it will become possible to attack more general varieties. This is a very difficult problem and we still don't have any clue about how to do that nevertheless it will certainly come.

There are various trick that use abelian varieties in order to deduce other Diophantine results. Nevertheless those tricks are not truly satisfying and in order to have serious Diophantine results one will have to attack more general geometrical objects.

The perspective of this is incredible because those questions deal with the core of what numbers are and how they arrange together. It may be a little bit futurist but we can truly hope that with such Mathematics we will be able to enter this very deeply and to have some serious understanding of Number Theory.

As it has stayed for almost two thousand years as almost unsolvable questions, this is a part of Mathematics that has not developed so much till now and actually there are only very few results and methods. However we can truly expect that this will develop thanks to those new tools and techniques and that things will progressively appear to be easier to deal with.

Number Theory is from logical and structural points of view extremely rigid. It is almost impossible to invent or create anything in Number Theory because of this rigidity. What is clear with this geometric point of view is that something new is beginning to be revealed. Things will stay as rigid as before but it seems that it will be possible to have a much better understanding.

All Mathematics have begun around geometry and numbers, Number Theory itself has developed almost integrally around Diophantine questions, we can expect in the centuries that are coming a kind of revolution in Number Theory that will have a major repercussion on Mathematics.