Descartes vs Grothendieck

By Benjamin Wagener, July 29, 2014.

For having spent many years working with Algebraic Geometry and especially with Arithmetic and Diophantine Geometry, I would like to clarify something related to this kind of Geometry.

The development of elementary geometry; polygons, circles, conics…; has undergone a revolution with the introduction by René Descartes of coordinate systems. After this a new era in Mathematics has emerged were almost everything was seen represented by coordinates. This has culminated with the development of the related analysis and the development of manifold-like mathematics with the introduction of Manifolds, of Differential Varieties, of Riemannian Geometry which is a kind of “glued-together” coordinate systems with their naturally associated objects.

With this in hand it has become possible to explore various kinds of geometric objects and of various associated objects keeping this mental representation given by a feeling of “coordinates” as precise position in space.

In the late 1950’s, Alexander Grothendieck equipped with the proper tools providing the underlying structure of those Mathematics has built a completely new theory when dealing with algebraic objects, that is to say roughly objects given by polynomial equations, based upon the structure that a geometry should satisfies.

What is not clear for many is that this “Algebraic Geometry” goes much further than the spatial representation of René Descartes. The theory of Alexander Grothendieck has made Geometry a well founded Mathematics theory in the sense that he has made Geometry axiomatic. This way Alexander Grothendieck has made clear what we call Geometry.

This is especially obvious in is book “Le Langage des Schémas” (The Language of Schemes) where he develop the foundations of his corresponding Geometry. The title itself uses the word “Language” and its full meaning appears from the fact that he develops a quite axiomatic theory of the Geometric structures related to Algebra.

What I would like to explain here is that this theory is much more than a spatial representation. This theory reveals the true logical and structural meaning of algebraic objects. This geometry is not imposed this is a completely natural geometry that appears within algebra itself.

With respect to Arithmetic Geometry, Diophantine Geometry and Number Theory this is of no comparable usefulness.

Because schemes, for example, encapsulate the inner logical structure of the algebraic objects of study.

I would like to insist on this. When dealing with schemes and algebraic geometry , one doesn't deal truly with spatial representation of objects like with the geometry of Descartes, one deal with their inner "architecture".

This is mainly why one can expect so much from Arithmetic Geometry in Number Theory.

One sometime forgets this or is not fully aware that when dealing with Algebraic Geometry, it is no longer spatial, it is architecture. The full structures of algebraic objects emerge with the associated tool to work with them.

From this point of view the Geometry of Alexander Grothendieck is much deeper than René Descartes' one but of course without this last it would have never appear.

It is especially astonishing to see that nowadays Geometry has a well defined meaning and that one can go that far in. The language we use for Geometry in the 21st century is understood in all the Mathematics community, whether it is in Topology, Differential Geometry, Analytic Geometric, Algebraic Geometry, Complex Geometry, Tropical Geometry, Non-Commutative Geometry,.., but what is especially interesting are:

-Globally for any kind of Geometry we have the proper tools to decipher the underlying architecture of objects.

-With respect to Algebraic Geometry it is more than amazing that a geometric theory emerges so naturally and is so well suited for algebraic objects. This is absolutely remarkable.